# Topological properties defined in terms of generalized open sets*

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**Abstract**

This paper covers some recent progress in the study of sg-open sets, sg-compact spaces, N-scattered spaces and some related concepts. A subset $A$ of a topological space $(X, \tau)$ is called sg-closed if the semi-closure of $A$ is included in every semi-open superset of $A$. Complements of sg-closed sets are called sg-open. A topological space $(X, \tau)$ is called sg-compact if every cover of $X$ by sg-open sets has a finite subcover. N-scattered space is a topological spaces in which every nowhere dense subset is scattered.

## 1 Prelude

Major part of the talk I presented in August 1997 at the Topological Conference in Yatsushiro College of Technology is based on the following three papers:

• J. Dontchev and H. Maki, On sg-closed sets and semi-$\lambda$-closed sets, *Questions Answers Gen. Topology*, (Osaka, Japan), **15** (2) (1997), to appear.

• J. Dontchev and M. Ganster, More on sg-compact spaces, *Portugal. Math.*, **55** (1998), to appear.

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• J. Dontchev and D. Rose, On spaces whose nowhere dense subsets are scattered, *Internat. J. Math. Math. Sci.*, to appear.

The paper in these Proceedings is a collection of the results from the above mentioned papers and of few new ideas and questions.

## 2 Sg-open sets and sg-compact spaces

In 1995, sg-compact spaces were introduced independently by Caldas [2] and by Devi, Balachandran and Maki [4]. A topological space $(X, \tau)$ is called *sg-compact* if every cover of $X$ by sg-open sets has a finite subcover.

Sg-closed and sg-open sets were introduced for the first time by Bhattacharyya and Lahiri in 1987 [1]. Recall that a subset $A$ of a topological space $(X, \tau)$ is called *sg-open* [1] if every semi-closed subset of $A$ is included in the semi-interior of $A$. A set $A$ is called *semi-open* if $A \subseteq \overline{\text{Int}A}$ and *semi-closed* if $\text{Int}\overline{A} \subseteq A$. The *semi-interior* of $A$, denoted by sInt$(A)$, is the union of all semi-open subsets of $A$, while the *semi-closure* of $A$, denoted by sCl$(A)$, is the intersection of all semi-closed supersets of $A$. It is well known that $\text{sInt}(A) = A \cap \overline{\text{Int}A}$ and $\text{sCl}(A) = A \cup \text{Int}\overline{A}$.

Sg-closed sets have been extensively studied in recent years mainly by (in alphabetical order) Balachandran, Caldas, Devi, Dontchev, Ganster, Maki, Noiri and Sundaram (see the references).

In the article [1], where sg-closed sets were introduced for the first time, Bhattacharyya and Lahiri showed that the union of two sg-closed sets is not in general sg-closed. On its behalf, this was rather an unexpected result, since most classes of generalized closed sets are closed under finite unions. Recently, it was proved [5, Dontchev; 1997] that the class of sg-closed sets is properly placed between the classes of semi-closed and semi-preclosed (= $\beta$-closed) sets. All that inclines to show that the behavior of sg-closed sets is more like the behavior of semi-open, preopen and semi-preopen sets than the one of 'generalized closed' sets (g-closed, gsp-closed, $\theta$-closed etc.). Thus, one is more likely to expect that arbitrary intersection of sg-closed sets is a sg-closed set. Indeed, in 1997 Dontchev and Maki [8] solved the first problem of Bhattacharyya and Lahiri in the positive.

**Theorem.** [8, Dontchev and Maki; 1997]. An arbitrary intersection of sg-closed sets is sg-closed.

Every topological space $(X, \tau)$ has a unique decomposition into two sets $X_1$ and $X_2$, where $X_1 = \{x \in X \colon \{x\}$ is nowhere dense$\}$ and $X_2 = \{x \in X \colon \{x\}$ is locally dense$\}$. This decomposition is due to Janković and Reilly [19]. Recall that a set $A$ is said to be *locally dense* [3, Corson and Michael; 1964] (= *preopen*) if $A \subseteq \mathrm{Int}\overline{A}$.

It is a fact that a subset $A$ of $X$ is sg-closed (= its complement is sg-open) if and only if $X_1 \cap \mathrm{sCl}(A) \subseteq A$ [8, Dontchev and Maki; 1997], or equivalently if and only if $X_1 \cap \mathrm{Int}\overline{A} \subseteq A$. By taking complements one easily observes that $A$ is sg-open if and only if $A \cap X_1 \subseteq \mathrm{sInt}(A)$. Hence every subset of $X_2$ is sg-open.

Next we consider the bitopological case and utterly $(\tau_i, \tau_j)$-Baire spaces:

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_i, \tau_j)$*-sg-closed* if $\tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) \subseteq U$ whenever $A \subseteq U$, $U \in SO(X, \tau_i)$ and $i, j \in \{1, 2\}$. Clearly every $(\tau_i, \tau_j)$-rare (= nowhere dense) set is $(\tau_i, \tau_j)$-sg-closed but not vice versa. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_i, \tau_j)$*-rare* [13, Fukutake, 1992] if $\tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) = \emptyset$, where $i, j \in \{1, 2\}$. $A$ is called $(\tau_i, \tau_j)$*-meager* [14, Fukutake 1992] if $A$ is a countable union of $(\tau_i, \tau_j)$-rare sets.

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_i, \tau_j)$*-sg-meager* if $A$ is a countable union of $(\tau_i, \tau_j)$-sg-closed sets. Clearly, every $(\tau_i, \tau_j)$-meager set is $(\tau_i, \tau_j)$-sg-meager but not vice versa.

**Definition.** [13, Fukutake; 1992]. A bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_i, \tau_j)$*-Baire* if $(\tau_i, \tau_j)$-$\mathcal{M} \cap \tau_i = \{\emptyset\}$, where $i, j \in \{1, 2\}$.

**Definition.** A bitopological space $(X, \tau_1, \tau_2)$ is called *utterly* $(\tau_i, \tau_j)$*-Baire* if $(\tau_i, \tau_j)$-sg-$\mathcal{M} \cap \tau_i = \{\emptyset\}$, where $i, j \in \{1, 2\}$.

Clearly every utterly $(\tau_i, \tau_j)$-Baire space is a $(\tau_i, \tau_j)$-Baire space but not conversely.

**Question 1.** How are utterly $(\tau_i, \tau_j)$-Baire space and $(\tau_i, \tau_j)$-semi-Baire spaces related? The class of $(\tau_i, \tau_j)$-semi-Baire spaces was introduced by Fukutake in 1996 [14]. Under what conditions is a $(\tau_i, \tau_j)$-Baire space utterly $(\tau_i, \tau_j)$-Baire?

**Question 2.** Let $(X, \tau_1, \tau_2)$ be a bitopological space such that $\tau_1 \subseteq \tau_2$ and $\tau_2$ is metrizable and complete. Under what additional conditions is $(X, \tau_1, \tau_2)$ an utterly $(\tau_i, \tau_j)$-Baire space? Note that $(X, \tau_1, \tau_2)$ is always a $(\tau_i, \tau_j)$-Baire space [14, Fukutake; 1992].

Observe that sg-open and preopen sets are concepts independent from each other.

**Theorem.** [22, Maki, Umehara, Noiri; 1996]. Every topological space is pre-$T_{\frac{1}{2}}$.

**Theorem.** Every topological space is sg-$T_{\frac{1}{2}}$, i.e., every singleton is either sg-open or sg-closed.

Improved Janković-Reilly Decomposition Theorem. Every topological space $(X, \tau)$ has a unique decomposition into two sets $X_1$ and $X_2$, where $X_1 = \{x \in X \colon \{x\}$ is nowhere dense$\}$ and $X_2 = \{x \in X \colon \{x\}$ is sg-open and locally dense$\}$.

Let $A$ be a sg-closed subset of a topological space $(X, \tau)$. If every subset of $A$ is also sg-closed in $(X, \tau)$, then $A$ will be called *hereditarily sg-closed* (= hsg-closed) [6]. Hereditarily sg-open sets are defined in a similar fashion. Observe that every nowhere dense subset is hsg-closed but not vice versa.

**Theorem.** [6, Dontchev and Ganster; 1998]. For a subset $A$ of a topological space $(X, \tau)$ the following conditions are equivalent:

(1) $A$ is hsg-closed.

(2) $X_1 \cap \mathrm{Int}\overline{A} = \emptyset$.

A topological space $(X, \tau)$ is called a *$C_2$-space* [15, Ganster; 1987] (resp. *$C_3$-space* [6]) if every nowhere dense (resp. hsg-closed) set is finite. Clearly every $C_3$-space is a $C_2$-space. Also, a topological space $(X, \tau)$ is indiscrete if and only if every subset of $X$ is hsg-closed (since in that case $X_1 = \emptyset$).

Semi-normal spaces can be characterized via sg-closed sets as follows:

**Theorem.** [24, Noiri; 1994]. A topological space $(X, \tau)$ is semi-normal if and only for each pair of disjoint semi-closed sets $A$ and $B$, there exist disjoint sg-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Question 3.** How do hsg-open sets characterize properties related to semi-normality?

In terms of sg-closed sets, pre sg-continuous functions and pre sg-closed functions were defined and investigated by Noiri in 1994 [24].

Following Hodel [20], we say that a *cellular family* in a topological space $(X, \tau)$ is a collection of nonempty, pairwise disjoint open sets. The following result reveals an interesting property of $C_2$-spaces.

**Theorem.** [6, Dontchev and Ganster; 1998]. Let $(X, \tau)$ be a $C_2$-space. Then, every infinite cellular family has an infinite subfamily whose union is contained in $X_2$.

The *$\alpha$-topology* [23, Njåstad; 1965] on a topological space $(X, \tau)$ is the collection of all sets of the form $U \setminus N$, where $U \in \tau$ and $N$ is nowhere dense in $(X, \tau)$. Recall that topological spaces whose $\alpha$-topologies are hereditarily compact have been shown to be *semi-compact* [17, Ganster, Janković and Reilly, 1990]. The original definition of semi-compactness is in terms of semi-open sets and is due to Dorsett [11]. By definition a topological space $(X, \tau)$ is called *semi-compact* if every cover of $X$ by semi-open sets has a finite subcover.

Remark. (i) The 1-point-compactification of an infinite discrete space is a $C_2$-space having an infinite cellular family.

(ii) [15, Ganster; 1987] A topological space $(X, \tau)$ is semi-compact if and only if $X$ is a $C_2$-space and every cellular family is finite.

(iii) [18, Hanna and Dorsett; 1984] Every subspace of a semi-compact space is semi-compact (as a subspace).

**Theorem.** [6, Dontchev and Ganster; 1998]. (i) Every $C_3$-space $(X, \tau)$ is semi-compact.

(ii) Every sg-compact space is semi-compact.

Remark. (i) It is known that sg-open sets are $\beta$-open, i.e. they are dense in some regular closed subspace. Note that $\beta$-compact spaces, i.e. the spaces in which every cover by $\beta$-open sets has a finite subcover are finite [16, Ganster, 1992]. However, one can easily find an example of an infinite sg-compact space – the real line with the cofinite topology is such a space.

(ii) In semi-$T_D$-spaces the concepts of sg-compactness and semi-compactness coincide. Recall that a topological space $(X, \tau)$ is called a *semi-$T_D$-space* [19, Janković and Reilly; 1985] if each singleton is either open or nowhere dense, i.e. if every sg-closed set is semi-closed.

**Theorem.** [6, Dontchev and Ganster; 1998]. For a topological space $(X, \tau)$ the following conditions are equivalent:

(1) $X$ is sg-compact.

(2) $X$ is a $C_3$-space.

Remark. (i) If $X_1 = X$, then $(X, \tau)$ is sg-compact if and only if $(X, \tau)$ is semi-compact. Observe that in this case sg-closedness and semi-closedness coincide.

(ii) Every infinite set endowed with the cofinite topology is (hereditarily) sg-compact.

As mentioned before, an arbitrary intersection of sg-closed sets is also a sg-closed set [8, Dontchev and Maki; 1997]. The following result provides an answer to the question about the additivity of sg-closed sets.

**Theorem.** [6, Dontchev and Ganster; 1998]. (i) If $A$ is sg-closed and $B$ is closed, then $A \cup B$ is also sg-closed.

(ii) The intersection of a sg-open and an open set is always sg-open.

(iii) The union of a sg-closed and a semi-closed set need not be sg-closed, in particular, even finite union of sg-closed sets need not be sg-closed.

**Problem.** Characterize the spaces, where finite union of sg-closed sets is sg-closed, i.e. the spaces $(X, \tau)$ for which $SGO(X, \tau)$ is a topology. Note: It is known that the spaces where $SO(X, \tau)$ is a topology is precisely the class of extremally disconnected spaces, i.e., the spaces where each regular open set is regular closed.

A result of Bhattacharyya and Lahiri from 1987 [1] states that if $B \subseteq A \subseteq (X, \tau)$ and $A$ is open and sg-closed, then $B$ is sg-closed in the subspace $A$ if and only if $B$ is sg-closed in $X$. Since a subset is regular open if and only if it is $\alpha$-open and sg-closed [9, Dontchev and Przemski; 1996], we obtain the following result:

**Theorem.** [6, Dontchev and Ganster; 1998]. Let $R$ be a regular open subset of a topological space $(X, \tau)$. If $A \subseteq R$ and $A$ is sg-open in $(R, \tau|R)$, then $A$ is sg-open in $X$.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called *$\delta$-open* [25, Veličko; 1968] if $A$ is a union of regular open sets. The collection of all $\delta$-open subsets of a topological space $(X, \tau)$ forms the so called *semi-regularization topology*.

**Corollary**. If $A \subseteq B \subseteq (X, \tau)$ such that $B$ is $\delta$-open in $X$ and $A$ is sg-open in $B$, then $A$ is sg-open in $X$.

**Theorem.** [6, Dontchev and Ganster; 1998]. Every $\delta$-open subset of a sg-compact space $(X, \tau)$ is sg-compact, in particular, sg-compactness is hereditary with respect to regular open sets.

Example. Let $A$ be an infinite set with $p \notin A$. Let $X = A \cup \{p\}$ and $\tau = \{\emptyset, A, X\}$.

(i) Clearly, $X_1 = \{p\}$, $X_2 = A$ and for each infinite $B \subseteq X$, we have $\overline{B} = X$. Hence $X_1 \cap \text{Int}\overline{B} \neq \emptyset$, so $B$ is not hsg-closed. Thus $(X, \tau)$ is a $C_3$-space, so sg-compact. But the open subspace $A$ is an infinite indiscrete space which is not sg-compact. This shows that hereditary sg-compactness is a strictly stronger concept than sg-compactness and '$\delta$-open' cannot be replaced with 'open'.

(ii) Observe that $X \times X$ contains an infinite nowhere dense subset, namely $X \times X \setminus A \times A$. This shows that even the finite product of two sg-compact spaces need not be sg-compact, not even a $C_2$-space.

(iii) [21, Maki, Balachandran and Devi; 1996] If the nonempty product of two spaces is sg-compact $T_{gs}$-space, then each factor space is sg-compact.

Recall that a function $f\colon (X, \tau) \to (Y, \sigma)$ is called *pre-sg-continuous* [24, Noiri, 1994] if $f^{-1}(F)$ is sg-closed in $X$ for every semi-closed subset $F \subseteq Y$.

**Theorem.** [6, Dontchev and Ganster; 1998]. (i) The property 'sg-compact' is topological.

(ii) Pre-sg-continuous images of sg-compact spaces are semi-compact.

# 3 N-scattered spaces

A topological space $(X, \tau)$ is *scattered* if every nonempty subset of $X$ has an isolated point, i.e. if $X$ has no nonempty dense-in-itself subspace. If $\tau^\alpha = \tau$, then $X$ is said to be an *$\alpha$-space* [10, Dontchev and Rose; 1996] or a *nodec space*. All submaximal and all globally disconnected spaces are examples of $\alpha$-spaces. Recall that a space $X$ is *submaximal* if every dense set is open and *globally disconnected* [12, El'kin; 1969] if every set which can be placed between an open set and its closure is open, i.e. if every semi-open set is open.

Recently $\alpha$-scattered spaces (= spaces whose $\alpha$-topologies are scattered) were considered by Dontchev, Ganster and Rose [7] and it was proved that a space $X$ is scattered if and only if $X$ is $\alpha$-scattered and N-scattered.

Recall that a topological ideal $\mathcal{I}$, i.e. a nonempty collection of sets of a space $(X, \tau)$ closed under heredity and finite additivity, is *$\tau$-local* if $\mathcal{I}$ contains all subsets of $X$ which are locally in $\mathcal{I}$, where a subset $A$ is said to be locally in $\mathcal{I}$ if it has an open cover each member of which intersects $A$ in an ideal amount, i.e. each point of $A$ has a neighborhood whose intersection with $A$ is a member of $\mathcal{I}$. This last condition is equivalent to $A$ being disjoint with $A^*(\mathcal{I})$, where $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau_x\}$ with $\tau_x$ being the open neighborhood system at a point $x \in X$.

**Definition.** [10, Dontchev and Rose; 1996]. A topological space $(X, \tau)$ is called *N-scattered* if every nowhere dense subset of $X$ is scattered.

Clearly every scattered and every $\alpha$-space, i.e. nodec space, is N-scattered. In particular, all submaximal spaces are N-scattered. The density topology on the real line is an example of an N-scattered space that is not scattered. The space $(\omega, L)$ below shows that even scattered spaces need not be $\alpha$-spaces. Another class of spaces that are N-scattered (but only along with the $T_0$ separation) is Ganster's class of $C_2$-spaces.

**Theorem.** [10, Dontchev and Rose; 1996]. If $(X, \tau)$ is a $T_1$ dense-in-itself space, then $X$ is N-scattered $\Leftrightarrow N(\tau) = S(\tau)$, where $N(\tau)$ is the ideal of nowhere dense subsets of $X$, and $S(\tau)$ is the ideal of scattered subsets of $X$.

Example. Let $X = \omega$ have the cofinite topology $\tau$. Then $X$ is a $T_1$ dense-in-itself space with $N(\tau) = I_\omega$, where $I_\omega$ is the ideal of all finite sets. Clearly, $X$ is an N-scattered space, since $N(\tau) = I_\omega \subseteq S(\tau)$. Note that $N(\tau) = S(\tau)$. Also, $X$ is far from being $(\alpha)$-scattered having no isolated points. It may also be observed that the space of this example is N-scattered being an $\alpha$-space.

Remark. A space $X$ is called (pointwise) homogenous if for any pair of points $x, y \in X$, there is a homeomorphism $h: X \to X$ with $h(x) = y$. Topological groups are such spaces. Further, such a space is either crowded or discrete. For if one isolated point exists, then all points are isolated. However, the above given space $X$ is a crowded homogenous N-scattered space.

Noticing that scatteredness and $\alpha$-scatteredness are finitely productive might suggest that N-scatteredness is finitely productive. But this is not the case.

Example. The usual space of Reals, $(R, \mu)$ is rim-scattered but not N-scattered. Certainly, the usual base of bounded open intervals has the property that nonempty boundaries of its members are scattered. However, the nowhere dense Cantor set is dense-in-itself. Another example of a rim-scattered space which is not N-scattered is constructed by Dontchev, Ganster and Rose [7].

Remark. It appears that rim-scatteredness is much weaker that N-scatteredness.

**Theorem.** [10, Dontchev and Rose; 1996]. N-scatteredness is hereditary.

**Theorem.** [10, Dontchev and Rose; 1996]. The following are equivalent:

(a) The space $(X, \tau)$ is N-scattered.

(b) Every nonempty nowhere dense subspace contains an isolated point.

(c) Every nowhere dense subset is scattered, i.e., $N(\tau) \subseteq S(\tau)$.

(d) Every closed nowhere dense subset is scattered.

(e) Every nonempty open subset has a scattered boundary, i.e., $\mathrm{Bd}(U) \in S(\tau)$ for each $U \in \tau$.

(f) The $\tau^\alpha$-boundary of every $\alpha$-open set is $\tau$-scattered.

(g) The boundary of every nonempty semi-open set is scattered.

(h) There is a base for the topology consisting of N-scattered open subspaces.

(i) The space has an open cover of N-scattered subspaces.

(j) Every nonempty open subspace is N-scattered.

(k) Every nowhere dense subset is $\alpha$-scattered.

**Corollary**. Any union of open N-scattered subspaces of a space $X$ is an N-scattered subspace of $X$.

Remark. The union of all open N-scattered subsets of a space $(X, \tau)$ is the largest open N-scattered subset $NS(\tau)$. Its complement is closed and if nonempty contains a nonempty crowded nowhere dense set. Moreover, $X$ is N-scattered if and only if $NS(\tau) = X$. Since partition spaces are precisely those having no nonempty nowhere dense sets, such spaces are N-scattered. On the other hand we have the following chain of implications. The space $X$ is discrete $\Rightarrow$ $X$ is a partition space $\Rightarrow$ $X$ is zero dimensional $\Rightarrow$ $X$ is rim-scattered. Also, $X$ is globally disconnected $\Rightarrow$ $X$ is N-scattered. However, this also follows quickly

from the easy to show characterization $X$ is globally disconnected $\Leftrightarrow$ $X$ is an extremally disconnected $\alpha$-space, and the fact that every $\alpha$-space is N-scattered. Actually, something much stronger can be noted. Every $\alpha$-space is N-closed-and-discrete, i.e. $N(\tau) \subseteq CD(\tau)$. Of course, $CD(\tau) \subseteq D(\tau) \subseteq S(\tau)$, where $CD(\tau)$ is the ideal of closed and discrete subsets of $(X, \tau)$, and $D(\tau)$ is the family of all discrete sets. We will show later that for a non-N-scattered space $(X, \tau)$ in which $NS(\tau)$ contains a non-discrete nowhere dense set, $\tau^\alpha$ is not the smallest expansion of $\tau$ for which $X$ is N-scattered, i.e., there exists a topology $\sigma$ strictly intermediate to $\tau$ and $\tau^\alpha$ such that $NS(\sigma) = X$.

Local N-scatteredness is the same as N-scatteredness.

**Theorem.** [10, Dontchev and Rose; 1996]. If every point of a space $(X, \tau)$ has an N-scattered neighborhood, then $X$ itself is N-scattered.

In the absence of separation, a finite union of scattered sets may fail to be scattered. For example, the singleton subsets of a two-point indiscrete spaces are scattered. But given two disjoint scattered subsets, if one has an open neighborhood disjoint from the other, then their union is scattered.

**Theorem.** [10, Dontchev and Rose; 1996]. In every $T_0$-space $(X, \tau)$, finite sets are scattered, i.e., $I_\omega \subseteq S(\tau)$.

**Theorem.** [10, Dontchev and Rose; 1996]. Let $(X, \tau)$ be a non-N-scattered space, so that $NP(\tau) = X \setminus NS(\tau) \neq \emptyset$. Suppose also that $NS(\tau)$ contains a nonempty non-discrete nowhere dense subset. Then there is a topology $\sigma$ with $\tau \subset \sigma \subset \tau^\alpha$ such that $(X, \sigma)$ is N-scattered.

In search for a smallest expansion of $\sigma$ and $\tau$ for which $(X, \sigma)$ is N-scattered, we have the following:

**Theorem.** [10, Dontchev and Rose; 1996]. Let $(X, \tau)$ be a space and let $I = \{A \subseteq E \colon E$ is a perfect (closed and crowded) nowhere dense subset of $(X, \tau)\}$. Then $(X, \gamma)$ is N-scattered, where $\gamma = \tau[I]$, the smallest expansion of $\tau$ for which members of $I$ are closed.

**Theorem.** [10, Dontchev and Rose; 1996]. Every closed lower density topological space $(X, F, I, \phi)$ for which $I$ is a $\sigma$-ideal containing finite subsets of $X$ is N-scattered. Recall that a lower density space $(X, F, I, \phi)$ is closed if $\tau_\phi \subseteq F$.

**Corollary**. If $(X, F, I, \phi)$ is a closed lower density space and $I$ is a $\sigma$-ideal with $I_\omega \subseteq I$, then $I = S(\tau_\phi)$.

**Corollary**. The space of real numbers $R$ with the density topology $\tau_d$ is N-scattered, and moreover, the scattered subsets are precisely the Lebesgue null sets.

The following theorem holds and thus we have another (perhaps new) characterization of $T_0$ separation. A similar characterization holds for $T_1$ separation.

**Theorem.** [10, Dontchev and Rose; 1996]. A space $(X, \tau)$ has $T_0$ separation if and only if $I_\omega \subseteq S(\tau)$.

Here is the relation between $C_2$-spaces and $N$-scattered spaces:

**Corollary**. Every $C_2T_0$-space is N-scattered.

**Theorem.** [10, Dontchev and Rose; 1996]. A space $(X, \tau)$ has $T_1$ separation if and only if $I_\omega \subseteq D(\tau)$.

**Theorem.** [10, Dontchev and Rose; 1996]. If $(X, \tau)$ is a $T_0$-space and if $S$ is any scattered subset of $X$ and if $F$ is any finite subset of $X$, then $S \cup F$ is scattered.

**Corollary**. Every $T_0$-space which is the union of two scattered subspaces is scattered.

**Corollary**. The family of scattered subsets in a $T_0$-space is an ideal.

Example. Let $(X, <)$ be any totally ordered set. Then both the left ray and right ray topologies $L$ and $R$ respectively, are $T_0$ topologies. They are not $T_1$ if $|X| > 1$. In case $X = \omega$ with the usual ordinal ordering $<$, $L$ and $R$ are in fact $T_D$ topologies, i.e. singletons are locally closed. The space $(\omega, L)$, where proper open subsets are finite, is scattered. For if $\emptyset \neq A \subseteq \omega$ let $n$ be the least element of $A$. Then the open ray $[0, n+1) = [0, n]$ intersects $A$ only at $n$. Thus, $n$ is an isolated point of $A$. Evidently, $S(L) = P(\omega)$, the maximum ideal. However, $S(R) = I_\omega$, the ideal of finite subsets. For if $A$ is any infinite subset of $\omega$, $A$ is crowded. For if $m \in A$ and if $U$ is any right directed ray containing $m$, $(U \setminus \{m\}) \cap A \neq \emptyset$, since $U$ omits only finitely many points of $\omega$. But every finite subset is scattered. Of course $S(L)$ and $S(R)$ are ideals in the last two examples since both $L$ and $R$ are $T_D$ topologies.

Remark. Note that the space $(\omega, R)$ is a crowded $T_D$-space, which is the union of an increasing (countable) chain of scattered subsets. In particular, $\omega = \cup_{k \in \omega} \{n < k \colon k \in \omega\}$ and for each $k$, $\{n < k \colon k \in \omega\} \in I_\omega \subseteq S(R)$. This seems to indicate that it is not likely

that an induction argument on the cardinality of a scattered set $F$ can be used similar to the above argument to show that $S \cup F$ is scattered if $S$ is scattered.

**Question 4.** Characterize the spaces where every hsg-closed set is scattered. How are they related to other classes of generalized scattered spaces? Note that:

$$\text{Scattered} \Rightarrow \text{hsg-scattered} \Rightarrow N\text{-scattered}$$

Note that the real line with the cofinite topology is an example of a hsg-scattered space, which is not scattered, while the real line with the indiscrete topology provides an example of an $N$-scattered space that is not hsg-scattered.

More generally, if $\mathcal{I}$ is a topological (sub)ideal on a space $(X, \tau)$, investigate the class of $\mathcal{I}$-scattered spaces, i.e. the spaces satisfying the condition: "Every $I \in \mathcal{I}$ is a scattered subspace of $(X, \tau)$".

Note that:

$\mathcal{F}$-scattered $\Leftrightarrow$ $T_0$-space

$\mathcal{C}$-scattered $\Leftrightarrow$ ?

$\mathcal{N}$-scattered $\Leftrightarrow$ $N$-scattered space

$\mathcal{M}$-scattered $\Leftrightarrow$ ?

$\mathcal{P}(X)$-scattered $\Leftrightarrow$ Scattered space

Of course, every space is $\mathcal{CD}$-scattered.

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